LESSON

4 Trigonometry

Some very interesting and important functions are formed by dividing the length of one side of a right triangle by the length of another side. These functions are called *trigonometric* because they come from the geometry of a triangle. The domain consists of the measures x of angles. Let H represent the length of the *hypotenuse*, A represent the length of the side *adjacent* to the angle x, and the letter O represent the length of the side *opposite* (away) from the angle x. A right triangle with angle x is depicted in Figure 4.1.

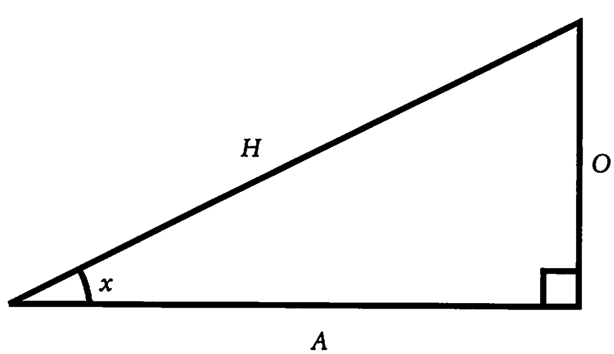

Figure 4.1

Mnemonic Hint

Some people remember the first three trigonometric functions by saying "Oliver Had A Heap Of Apples" to remember the $\frac{O}{H}$, $\frac{A}{H}$, and $\frac{O}{A}$ of sin(*x*), cos(*x*), and tan(*x*). Others say SOA CAH TOA to remember sin(*x*) = $\frac{O}{A}$, cos(*x*) = $\frac{A}{H}$, and tan(*x*) = $\frac{O}{A}$.

The six trigonometric functions, *sine* (abbreviated sin), *cosine* (cos), *tangent* (tan), *secant* (sec), *cosecant* (csc), and *cotangent* (cot), are defined for each angle x by dividing the following sides:

$$\sin(x) = \frac{O}{H}$$

$$\cos(x) = \frac{A}{H}$$

$$\tan(x) = \frac{O}{A}$$

$$\sec(x) = \frac{H}{A}$$

$$\csc(x) = \frac{H}{O}$$

$$\cot(x) = \frac{A}{O}$$

The first thing to notice is that all of the functions can be obtained from just $\sin(x)$ and $\cos(x)$ using the following *trigonometric identities.*

$$\tan(x) = \frac{O}{A} = \frac{\frac{O}{H}}{\frac{A}{H}} = \frac{\sin(x)}{\cos(x)}$$

$$\sec(x) = \frac{H}{A} = \frac{1}{\frac{A}{H}} = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{H}{O} = \frac{1}{\frac{O}{H}} = \frac{1}{\sin(x)}$$

$$\cot(x) = \frac{A}{O} = \frac{\frac{A}{H}}{\frac{O}{H}} = \frac{\cos(x)}{\sin(x)}$$

Thus, all of the trigonometric functions can be evaluated for an angle x if the $\sin(x)$ and $\cos(x)$ are known.

The next thing to notice is that the Pythagorean theorem, which, stated in terms of the sides O, A, and H, is $O^2 + A^2 = H^2$. And, if we divide everything by H^2, we get the following:

$$\frac{O^2}{H^2} + \frac{A^2}{H^2} = \frac{H^2}{H^2}$$

$$\left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2 = 1$$

Thus, $(\sin(x))^2 + (\cos(x))^2 = 1$. To save on parentheses, we often write this as $\sin^2(x) + \cos^2(x) = 1$. Because no particular value of x was used in the calculations, this is true for every value of x.

Drawing triangles and measuring their sides is an impractical and inaccurate method to calculate the values of trigonometric functions. Most people use calculators instead. Although, when using a calculator, it is very important to make sure that it is set to the same format for measuring angles that you are already using: that is, *degrees* or *radians.*

Conversion Hint

To convert from degrees to radians, multiply by $\frac{2\pi}{360} = \frac{\pi}{180}$.

To convert from radians to degrees, multiply by $\frac{360}{2\pi} = \frac{180}{\pi}$.

There are 360 degrees in a circle, possibly because ancient peoples thought that there were 360 days in a year. As the earth went around the sun, the position of the sun against the background stars moved one *degree* every day. The 2π radians in a circle correspond to the distance around a circle of radius 1. Because radians already correspond to a distance, there is no need for conversions when calculating with radians. Mathematicians thus use radians almost exclusively.

- To convert from degrees to radians, multiply by $\frac{2\pi}{360} = \frac{\pi}{180}$.
- To convert from radians to degrees, multiply by $\frac{360}{2\pi} = \frac{180}{\pi}$.

Example

Convert 45° into radians.

Solution

$45^\circ = 45 \cdot \frac{\pi}{180}$ radians $= \frac{\pi}{4}$ radians

Example

Convert $\frac{2\pi}{3}$ radians into degrees.

Solution

$\frac{2\pi}{3}$ radians $= \frac{2\pi}{3} \cdot \frac{180^\circ}{\pi} = 120^\circ$

▶ Practice

Convert the following into radians.

1. 30°

2. 180°

3. 270°

4. 300°

5. 135°

Convert the following into degrees.

6. $\frac{\pi}{3}$

7. $\frac{\pi}{2}$

8. 2π

9. $\frac{\pi}{10}$

10. $\frac{4\pi}{2}$

▶ Trigonometric Values of Nice Angles

There are a few nice angles for which the trigonometric functions can be easily calculated. If $x = \frac{\pi}{4} = 45^\circ$, then the two legs of the triangle are equal. If the hypotenuse is $H = 1$, then we have what you can see in Figure 4.2.

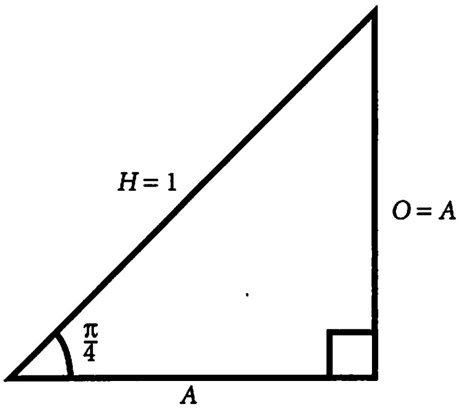

Figure 4.2

By the Pythagorean theorem, $A^2 + A^2 = 1$, so $2A^2 = 1$ and $A^2 = \frac{1}{2}$. This means that $O = A = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$. If we rationalize the denominator, we get $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Thus:

$$\sin\left(\frac{\pi}{4}\right) = \frac{O}{H} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{A}{H} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

Another nice angle is $x = 60^\circ = \frac{\pi}{3}$, because it is found in equilateral triangles such as those seen in Figure 4.3. This triangle can be cut in half to form the triangle shown in Figure 4.4.

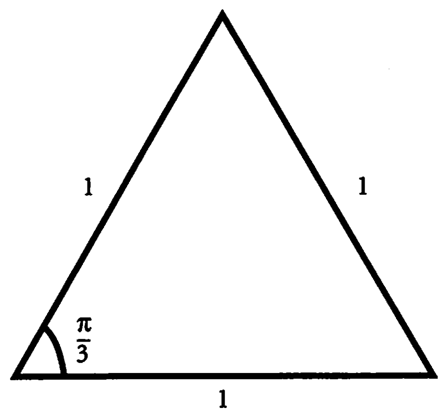

Figure 4.3

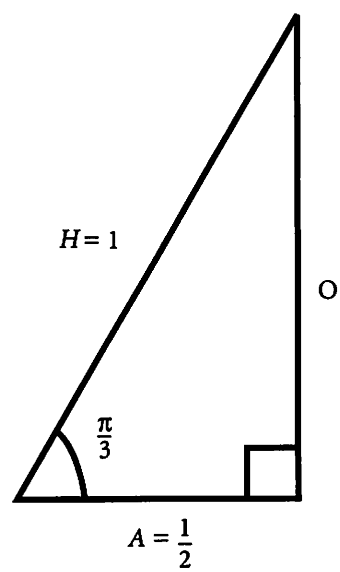

Figure 4.4

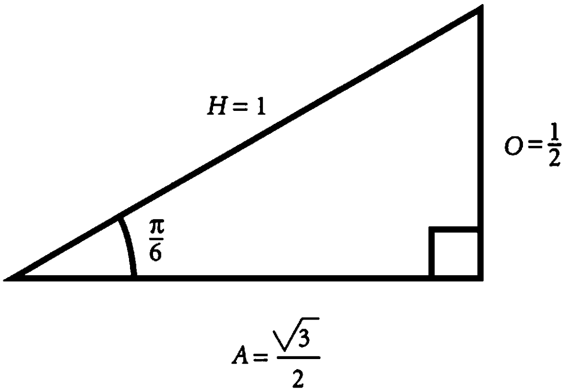

Figure 4.5

By the Pythagorean theorem, $\left(\frac{1}{2}\right)^2 + O^2 = 1^2$, so $O^2 = 1 - \frac{1}{4} = \frac{3}{4}$. Thus, $O = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$. This means that:

$$\sin\left(\frac{\pi}{3}\right) = \frac{O}{H} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{A}{H} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

We can flip that last triangle around to calculate the trigonometric functions for the other angle $x = 30^\circ = \frac{\pi}{6}$ (see Figure 4.5).

$$\sin\left(\frac{\pi}{6}\right) = \frac{O}{H} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{A}{H} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

Example

Use the trigonometric identities to find $\sec\left(\frac{\pi}{4}\right)$.

Solution

Use the trigonometric identity for sec.

$$\sec(x) = \frac{1}{\cos(x)}$$

Use $x = \frac{\pi}{4}$.

$$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos(\frac{\pi}{4})}$$

Use $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

$$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\frac{\sqrt{2}}{2}}$$

Simplify.

$$\sec\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

▶ Practice

Use the trigonometric identities to evaluate the following.

11. $\tan\left(\frac{\pi}{4}\right)$

12. $\tan\left(\frac{\pi}{3}\right)$

13. $\csc\left(\frac{\pi}{6}\right)$

14. $\sec\left(\frac{\pi}{3}\right)$

15. $\cot\left(\frac{\pi}{3}\right)$

16. $\cot\left(\frac{\pi}{6}\right)$

17. $\sec\left(\frac{\pi}{6}\right)$

18. $\csc\left(\frac{\pi}{4}\right)$

▶ Trigonometric Values for Angles Greater Than $90^\circ = \frac{\pi}{2}$

Notice that when the hypotenuse has length 1, the sine of the angle is the height of the triangle and the cosine is the width. Equivalently, the sine is the y-value of the point where a ray of the given angle intersects with the circle of radius 1. Similarly, the cosine is the x-value.

Example

For example, when $x = \frac{\pi}{6} = 30^\circ$, we have the picture shown in Figure 4.6.

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

The circle of radius 1 around the origin is called the *unit circle*. Note in Figure 4.7 that the angle of measure 0 runs straight to the right along the positive x-axis, and every other angle is measured counterclockwise up from this.

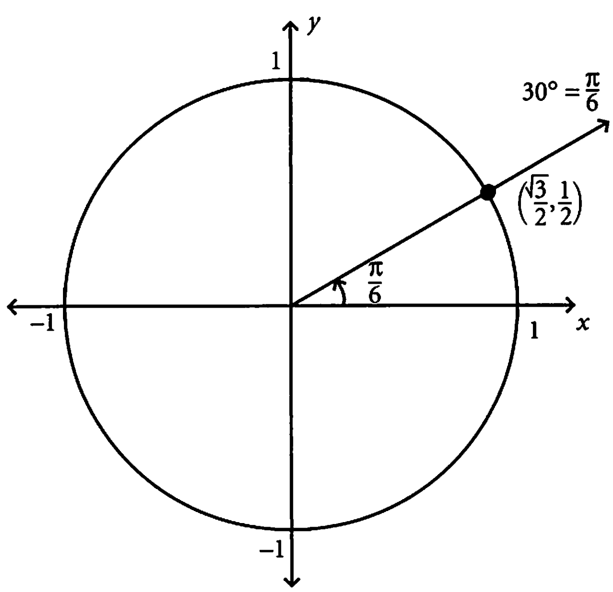

Figure 4.6

This can be used to find the trigonometric values of nice angles greater than $90^\circ = \frac{\pi}{2}$. The trick is to use either a $30^\circ, 60^\circ, 90^\circ$ triangle (a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle) or else a $45^\circ, 45^\circ, 90^\circ$ triangle (a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle) to find the y-value(sine) and x-value (cosine) of the appropriate point on the unit circle. As before, calculating the trigonometric values for non-nice angles requires the help of either much more mathematics or the use of a calculator.

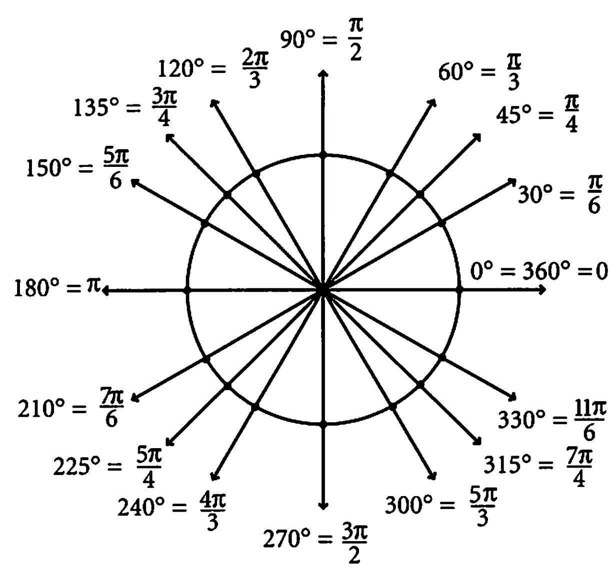

Figure 4.7

Example

Find the sine and cosine of $120^\circ = \frac{2\pi}{3}$.

Solution

For this angle, we use a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle, as shown in Figure 4.8 to find the x- and y-values. The y-value of the point where the ray of angle $\frac{2\pi}{3}$ hits

the unit circle is $y = \frac{\sqrt{3}}{2}$. Thus, $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

The x-value is negative, $x = -\frac{1}{2}$, so $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

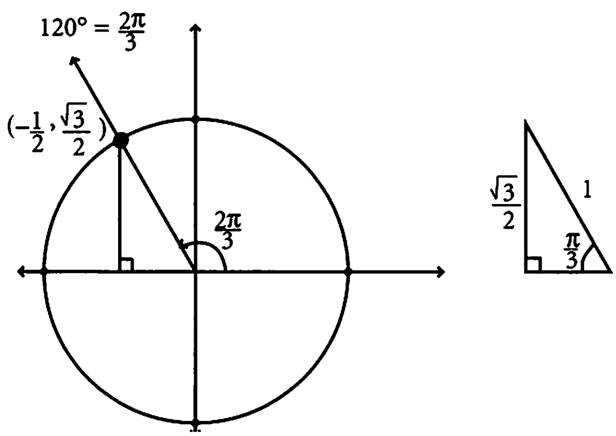

Figure 4.8

Example

Find the sine and cosine of $\frac{5\pi}{4} = 225°$.

Solution

Because $225°$ is a multiple of $45°$, we use a $45°, 45°, 90°$ triangle to find the x- and y-values. As seen in Figure 4.9, both of the coordinates are negative, so $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ are both negative.

Example

Find all of the trigonometric values for $90° = \frac{\pi}{2}$.

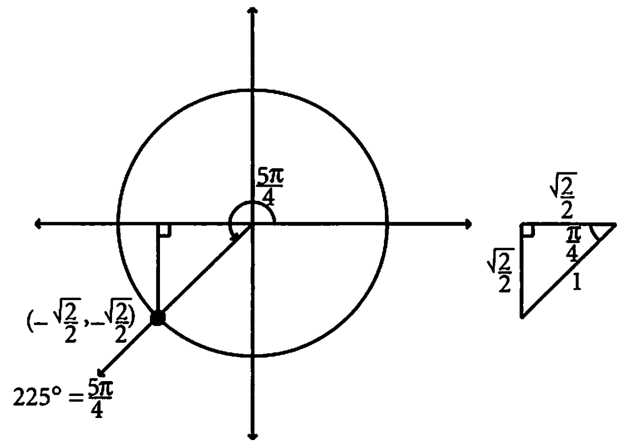

Figure 4.9

Solution

Even though there isn't a triangle here, there is still a point on the unit circle. See Figure 4.10. We conclude that $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$ from the x- and y-values of the point. Using the trigonometric identities, we can calculate that $\csc\left(\frac{\pi}{2}\right) = \frac{1}{\sin(\frac{\pi}{2})} = 1$ and $\cot\left(\frac{\pi}{2}\right) = \frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} = \frac{0}{1} = 0$. The tangent and secant

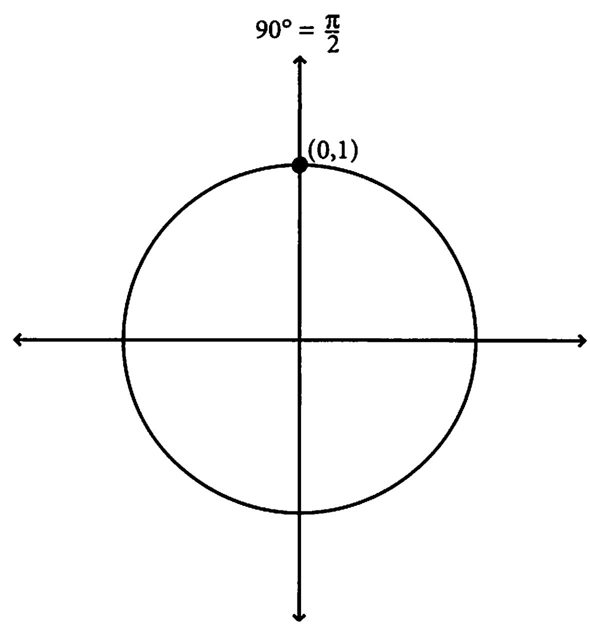

Figure 4.10

functions, however, involve division by 0 and thus are left undefined. The angle $x = \frac{\pi}{2}$ is not in the domain of tan and sec.

Notice that all of the trigonometric functions are the same at $0^\circ = 0$ and $360^\circ = 2\pi$. This is because turning around 360° leaves you facing in your original direction. Thus, everything repeats at this point.

Using the table along with the fact that everything repeats, we can sketch the graphs of $\sin(x)$ and $\cos(x)$. See Figures 4.11 and 4.12.

The functions sine and cosine are classic examples of repeating, or *oscillating*, functions because of the way they wave up and down forever.

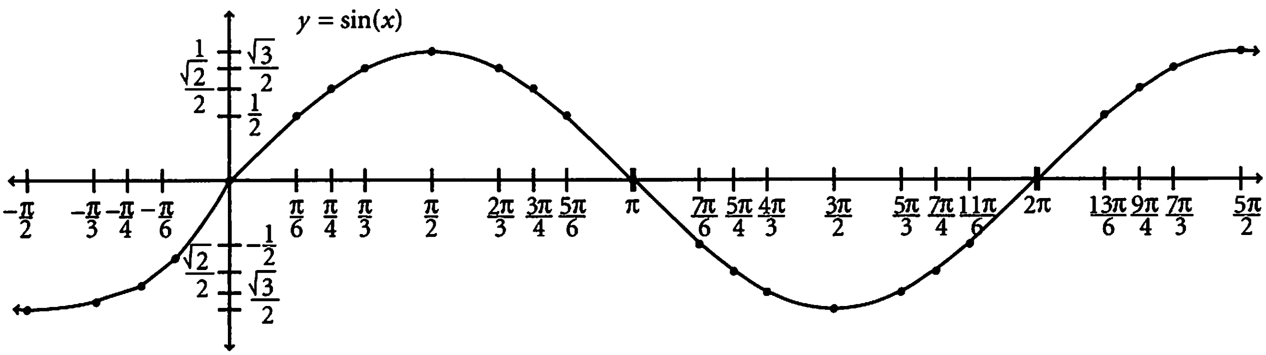

Figure 4.11

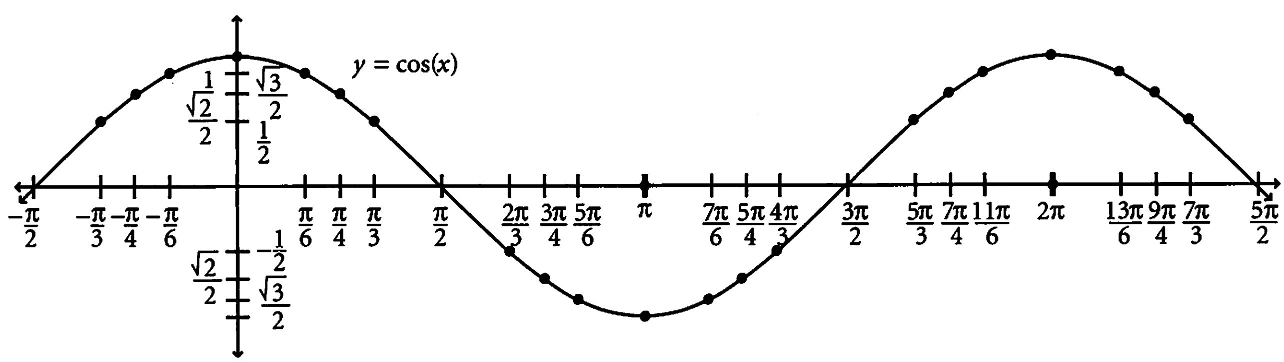

Figure 4.12

▶ Practice

Use the unit circle and the trigonometric identities to complete the following table. Find the answers to questions 19 through 36.

	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\sec(x)$	$\csc(x)$	$\cot(x)$
$0^\circ = 0$	0	1	0	1	undef.	undef.
$30^\circ = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	***19***	$\frac{2\sqrt{3}}{3}$	2	$\sqrt{3}$
$45^\circ = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60^\circ = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	2	$\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$
$90^\circ = \frac{\pi}{2}$	1	0	undef.	undef.	1	0
$120^\circ = \frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	-2	***20***	$-\frac{\sqrt{3}}{3}$
$135^\circ = \frac{3\pi}{4}$	***21***	***22***	***23***	***24***	***25***	***26***
$150^\circ = \frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\frac{2\sqrt{3}}{3}$	2	$-\sqrt{3}$
$180^\circ = \pi$	***27***	-1	0	-1	***28***	undef.
$210^\circ = \frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$-\frac{2\sqrt{3}}{3}$	-2	***29***
$225^\circ = \frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
$240^\circ = \frac{4\pi}{3}$	***30***	***31***	***32***	***33***	***34***	***35***
$270^\circ = \frac{3\pi}{2}$	-1	0	undef.	undef.	-1	0
$300^\circ = \frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	2	$-\frac{2\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3}$
$315^\circ = \frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	***36***	$-\sqrt{2}$	-1
$360^\circ = 2\pi$	0	1	0	1	undef.	undef.

Note: The numbers appearing in bold with asterisks are questions 19 through 36.

19. Find the value that goes in the position in the table where you see *19*.

20. Find the value that goes in the position in the table where you see *20*.

21. Find the value that goes in the position in the table where you see *21*.

22. Find the value that goes in the position in the table where you see *22*.

23. Find the value that goes in the position in the table where you see *23*.

24. Find the value that goes in the position in the table where you see *24*.

25. Find the value that goes in the position in the table where you see *25*.

26. Find the value that goes in the position in the table where you see *26*.

27. Find the value that goes in the position in the table where you see *27*.

28. Find the value that goes in the position in the table where you see *28*.

29. Find the value that goes in the position in the table where you see *29*.

30. Find the value that goes in the position in the table where you see *30*.

31. Find the value that goes in the position in the table where you see *31*.

32. Find the value that goes in the position in the table where you see *32*.

33. Find the value that goes in the position in the table where you see *33*.

34. Find the value that goes in the position in the table where you see *34*.

35. Find the value that goes in the position in the table where you see *35*.

36. Find the value that goes in the position in the table where you see *36*.